1. (1) Please verify that “the method of variation of parameters” also applies to Euler’s Equation. (15%)

\[ y'' + \frac{1}{x} Ay' + \frac{1}{x^2} By = f(x) \]

(2) Use the result from (1) and solve the following differential equation. (10%)

\[ x^2y'' - 3xy' + 3y = 2x^3 + x^2 + 1 \]

2. Matrix \( [A] = \begin{bmatrix} a & 15 & -2 \\ 15 & b & d \\ -2 & d & c \end{bmatrix} \) has eigenvectors \( E_1, E_2, E_3 \)

\[ [E_1] = \begin{bmatrix} 1 \\ 2 \\ x_1 \end{bmatrix}, [E_2] = \begin{bmatrix} x_2 \\ 1 \\ 3 \end{bmatrix}, [E_3] = \begin{bmatrix} 4 \\ x_3 \\ -1 \end{bmatrix}, \text{is it possible to find } x_1, x_2 \text{ and } x_3? \]

Please explain. (15%)

3. Please use the definition of Laplace Transform to find

(1) \( \mathcal{L}[\sin(at)] \) (5%)

(2) \( \mathcal{L}[e^{2t} + 2t^2e^{-3t}] \) (5%)
4. 已知一微分方程式為

\[
\frac{d^2 y(x)}{dx^2} + a \frac{dy(x)}{dx} + y(x) = \sin(bx), \quad x \geq 0
\]

其中 \(-2 < a < 2\); \(b > 0\); 且 \(y(0) = 0\), \(\frac{dy(0)}{dx} = 0\)

(1) 若 \(a = 1\) 且 \(b = 2\)，求此微分方程式之全解。 (15%)

(2) 若需根據 \(a\) 及 \(b\) 之所有可能數值，討論如何求解此微分方程式，請問需考慮哪幾種情況？ (5%)

5. (1) 求 \(\sin \left( \frac{x}{2} \right)\) 之傅立葉級數展開式 (Fourier Series Representation)。 (15%)

(2) 根據(1)之結果求無窮級數

\[
\frac{-1}{4 \times 1^2 - 1} + \frac{1}{4 \times 2^2 - 1} + \frac{-1}{4 \times 3^2 - 1} + \frac{1}{4 \times 4^2 - 1} + \cdots
\]

之值。 (5%)

6. 某一質點在空間中移動，其位置向量為 \(\bar{S}(t) = (t^3 - t)\hat{i} + (2t^2 + t)\hat{j} + (3t)\hat{k}\)，其中 \(t(\geq 0)\) 是時間，求此質點在 \(t = 1\) 沿切線方向之加速度向量及沿法線方向之加速度向量。 (10%)