1. (8%) 90% of new airport-security personnel have had prior training in weapon detection. During their first month on the job, personnel without prior training fail to detect a weapon 3% of the time, while those with prior training fail only 0.5% of the time. What is the probability a new airport-security employee, who fails to detect a weapon during the first month on the job, has had prior training in weapon detection?

2. (7%) For all \( n \in \mathbb{Z}, n \geq 0 \), prove that \( n^3 + (n+1)^3 + (n+2)^3 \) is divisible by 9.

3. (15%) Let an alphabet \( \Sigma = \{0, 1\} \) and a language \( A = \{0, 01, 011, 0111, 1111\} \subseteq \Sigma^* \). For \( n \geq 1 \), let \( a_n \) count the number of strings in the Kleene closure \( A^* \) of length \( n \).
   (a) Find a recurrence relation for \( a_n \). (7%)
   (b) Solve the recurrence relation in (a). (8%)

4. (10%)
   (a) Let \( T = (V, E) \) be a tree with \( |V| = n \geq 2 \). How many distinct paths are there in the tree, \( T^* \)? (5%)
   (b) A tree has three vertices of degree 2, two vertices of degree 3, and four vertices of degree 4. How many vertices of degree 1 does it have? (5%)

5. (10%) In how many ways can 16 different books be distributed among four children so that
   (a) each child gets four books? (5%)
   (b) the two oldest children get five books and the two youngest get three books each? (5%)
6. (10%) Suppose that $A = SAS^{-1}$, where $A$ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $S$ is an $n \times n$ matrix with columns $s_i$, for $i = 1, 2, \ldots, n$.

(a) Show that if $x = \sum_{i=1}^{n} \alpha_i s_i$, then $A^k x = \sum_{i=1}^{n} \alpha_i \lambda_i^k s_i$, where $k$ is a positive integer. (3%) 

(b) Suppose that $|\lambda_i| < 1$ for $i = 1, 2, \ldots, n$. What happens to $A^k x$ as $k \to \infty$? Explain. (5%) 

7. (10%) The linear transformation $T$ is defined by $T(x) = Ax$ where

$$A = \begin{bmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$ 

Find the kernel of $T$. 

8. (15%) Determine whether each statement is true or false. If the statement is true, prove it. Otherwise provide a counterexample.

(a) Let $A$ and $B$ be $n \times n$ matrices and $A$ is similar to $B$. If $A$ is a nonsingular matrix then $B$ is nonsingular too. (5%) 

(b) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of linearly dependent vectors in $V$, and let $T$ be a linear transformation from $V$ to $V$. Then the set $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly dependent. (5%) 

(c) Let $A$ be an $n \times n$ matrix. If the rank of $A$ is strictly less than $n$, then the system of linear equations $Ax = b$ has infinite many solutions for any vector $b \in \mathbb{R}^n$. (5%) 

9. (15%) If the $n \times n$ matrix $A$ can be written as the product of a lower triangular matrix $L$ and an upper triangular matrix $U$, then $A = LU$ is an LU-factorization of $A$.

(a) Find the LU-factorization of $A$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ -1 & -3 & 0 \end{bmatrix}.$$ 

(b) Let $A$ be the matrix given in (a) and $b = [-1, 0, 2]^T$. Solve the system of equations $Ax = b$ by the LU-factorization result from (a). (7%)