1. (24%) Which of the following statement(s) is/are correct? (2/3 for each)
   (a) If \( A \) is an \( n \times n \) matrix and \( A^2 = 0 \), then \( I_n + A \) is invertible, where \( I_n \) is the identity matrix.
   (b) There exists a 2 \( \times \) 2 matrix \( A \) such that the space of all matrices commuting with \( A \) is two dimensional.
   (c) If \( A \) is an \( n \times n \) matrix and \( A^T A = AA^T \), then \( A \) is orthogonal.
   (d) If \( A \) is an \( n \times n \) matrix, then \( \text{Ker}(A) = \text{Ker}(A^T A) \).
   (e) There is a 3 \( \times \) 3 matrix \( A \) such that \( A^3 = 0 \) and 1 is one of \( A \)'s eigenvalue.
   (f) Every two-dimensional subspace of \( \mathbb{R}^{2 \times 2} \) contains an invertible matrix.
   (g) There exists real invertible 4 \( \times \) 4 matrices \( A \) and \( P \) such that \( P^{-1} A P = 3 A \).
   (h) If \( A \) is a 2 \( \times \) 2 matrix and \( AA^T = A^2 \), then \( A \) is symmetric.

2. (15%) Let \( B_1 = \{ (1, 0, 1), (1, 1, 1), (-1, 0, 0) \} \) and \( B_2 = \{ (0, 3, -2), (-1, 10, -7), (1, -8, 6) \} \) be two bases of \( \mathbb{R}^3 \).
   (a) (5%) What is the matrix that converts \( B_1 \)-coordinates to \( B_2 \)-coordinates in \( \mathbb{R}^3 \)?
   (b) (10%) Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be a linear operator and \( [T]_{B_1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \). Find \( [T]_{B_2} \).

3. (11%) Let \( A = \begin{pmatrix} 4 & 3 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \end{pmatrix} \).
   (a) (5%) Find the eigenvalues of \( A \).
   (b) (6%) Find an orthogonal basis of \( A \)'s column space.

4. (10%) Which of the following statement(s) is/are correct? (2/3 for each)
   (a) \( ((p \lor r) \land (p \rightarrow q) \land (r \rightarrow q)) \rightarrow q \) is a tautology.
   (b) \( \forall x \exists y (\neg P(x, y)) \lor \forall x \exists y (\neg P(x, y)) \equiv \exists x (\forall y (\exists x \neg P(x, y))) \land \exists y (\forall x \neg P(x, y)) \).
   (c) \( \exists x \forall y (xy = x) \), where the domain for all variables consists of all integers.
   (d) Let \( A \) and \( B \) be sets. \( (B - A) \cup (\overline{A} \cup B) = (A \cup B) \cap B \).
   (e) Let \( A \) be the set of real numbers and \( B \) be the set of negative real numbers. Then \( A - B \) is countably infinite.

5. (5%) Let \( f(x) = \frac{1}{(x^2 + 1)} \) be a function from \( \mathbb{R} \) to \( \mathbb{R} \). Is \( f \) invertible, and if it is, what is its inverse?
6. (15%) Given a fair six-sided dice with values 1 to 6, please answer the following questions.

(a) (5%) What is the probability of getting exactly three times of odd numbers if we roll the dice ten times?

(b) (5%) How many times should we roll the dice to guarantee that at least a value appears more than five times?

(c) (5%) Suppose that we roll the dice 20 times and the value \( i \) is observed \( x_i \) times for \( i = 1, 2, 3, ..., 6 \). Please determine the number of possible solutions for \( x_i \). (Hint: For example, \( x_1 = 20, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0 \) is a possible solution representing the situation that the value of the dice is 1 for all 20 times.)

7. (20%) According to Figure 1 and Figure 2, please answer the following questions.

(a) (10%) Find the number of distinct paths of length three from \( A \) to \( C \) in the graph in Figure 1.

(b) (10%) Determine whether the graphs shown in Figure 2 are isomorphic or not and verify your answer.

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**Figure 1:**

![Graph 1](image1)

**Figure 2:**

![Graph 2](image2)